Constraint on Fuzzy Controller with Relaxing LMIs Conservatism

Hugang Han, Chun-Xiang Chen, and Toshijiro Tanaka

Abstract—This paper deals with stability analysis and control design for a class of nonlinear systems with input constraint when using T-S fuzzy model. After leading the control design to LMIs approach, the attention is paid to 1) how to easy the LMIs conditions so that the system is asymptotically stable in a so-called ellipsoid to be made as large as possible; 2) how to relax the LMIs conditions so that the LMIs problem is more feasible. Also a few examples are given to illustrate the theoretical analysis.

I. INTRODUCTION

In the T-S fuzzy model, local dynamics in different state-space regions are represented by linear models. The overall model of the system is obtained by fuzzy blending of these local models. The control design is carried out based on the fuzzy model by the so-called parallel distributed compensation (PDC) scheme [1]. For each local linear model, a linear feedback control is designed. The resulting overall controller is again a fuzzy blending of the individual linear controllers. Originally, Tanaka and his colleagues have provided some sufficient conditions for the stability of the T-S fuzzy systems in the sense of Lyapunov function of the local model [1]-[4]. The conditions for existence of common Lyapunov function are obtained by solving Linear Matrix Inequalities (LMIs).

In this paper, we consider a class of nonlinear systems with input constraint (saturation). Actually, system input saturation can severely degrade the closed-loop system performance and sometimes even make the otherwise stable closed-loop system unstable. Recently, a great attention has paid on the system with input saturation [10]-[12]. In general, input saturation problem is dealt with by either designing low gain control laws that, for a given bound on the initial conditions to avoid the saturation limits [3], or estimating the domain of attraction in the presence of input saturation [12]. In this paper, we focus our attention on the bounded initial conditions, and make an effort to ease the conditions.

On the other hand, The conditions for existence of common Lyapunov function for all local models in the LMIs approach sometimes, in fact, tend to be very conservative, because the common Lyapunov function may not exist at all for many systems, especially for those used to represent highly nonlinear complex system. Furthermore, here in this paper in order to deal with the input constraint there are some extra LMIs, it is more necessary to relax the LMIs conditions. To relax the LMIs conditions, Tanaka, Ikeda and Wang [4] utilized a fuzzy system’s property (3) and reported a pioneering work in this area. Kim and Lee [5] took account of the interactions among the subsystems and addressed them in a single matrix, as a result, the conservatism due to the interactions is reduced. Afterward, Teixeira, Assuncao, and Avellar [6] reported an extended version from Kim and Lee’s method. As far as we know, the method proposed by Teixeira et al. is most effective to relax the LMIs conditions.

In this paper, after leading the control design to LMIs approach, the attention is paid to 1) how to easy the LMIs conditions so that the system is asymptotically stable in a so-called ellipsoid to be made as large as possible; 2) how to relax the LMIs conditions so that the LMIs problem is more feasible. Also a few examples are given to illustrate the theoretical analysis.

II. T-S FUZZY MODEL

The T-S fuzzy model is described by fuzzy IF-THEN rules, which represent local linear input-output relations of a nonlinear system as follows [1].

Rule $i$: IF $x_1(t)$ is $M_1^i$ AND ... AND $x_n(t)$ is $M_n^i$ THEN $\dot{x}(t) = A_i x(t) + B_i u(t)$ (1)

where $i (= 1, 2, \ldots, r)$ represents $i$th fuzzy rule, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the system state, $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$ is the controller, constant matrices $A_i$ and $B_i$ are of appropriate dimensions, $M_j^i (j = 1, 2, \ldots, n)$ is the $j$th fuzzy set of the $i$th fuzzy rule with membership function $\mu_j^i(x_j(t)).$ Let

$$u_i(t) = \prod_{j=1}^{n} \mu_j^i(x_j(t))$$

Then, given a pair $(\dot{x}(t), u(t))$, the resulting fuzzy system model is inferred as the weighted average of the local models and has the form

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(t) (A_i x(t) + B_i u(t))$$ (2)

where for $i = 1, 2, \ldots, r$

$$\alpha_i(t) = \frac{w_i(t)}{\sum_{i=1}^{r} w_i(t)} \geq 0, \sum_{i=1}^{r} \alpha_i(t) = 1$$ (3)

Generally, the following PDC controller [1][2] can be designed for the T-S fuzzy model (2).

Control Rule $i$:

IF $x_1(t)$ is $M_1^i$ AND ... AND $x_n(t)$ is $M_n^i$ THEN $u(t) = -F_i x(t)$ (4)

where $F_i \in \mathbb{R}^{m \times n} (i = 1, 2, \ldots, r)$ is the controller matrix gain to be designed. The overall state feedback control law

Authors are all with the Department of Management Information System, Prefectural University of Hiroshima, Hiroshima, 734-8558 JAPAN (phone/fax: +81-82-251-8560; email: hhan@pu-hiroshima.ac.jp).
is finally represented as

\[ u(t) = -\sum_{i=1}^{r} \alpha_i F_i \mathbf{x}(t) \tag{5} \]

Then the closed-loop T-S fuzzy control system composed of (2) and (5) is described by

\[
\dot{\mathbf{x}}(t) = \sum_{i=1}^{r} \alpha_i (\mathbf{x}(t)^T \Lambda_{ii} \mathbf{x}(t) \\
+ 2 \sum_{i=1}^{r} \sum_{j>i}^{r} \alpha_i (\mathbf{x}(t)) \alpha_j (\mathbf{x}(t)) \Lambda_{ij} \mathbf{x}(t)) \tag{6}
\]

where \( G_{ij} = A_i - B_i F_j \), and \( \Lambda_{ij} = \frac{G_{ij} + G_{ji}}{2} \). Clearly, it holds that \( G_{ii} = \Lambda_{ii} \), and \( \Lambda_{ij} = \Lambda_{ji} \).

For the stability of control system (6), there is the following theorem that gives basic LMIs conditions [1].

**Theorem 1:** The equilibrium of the continuous fuzzy control system (6) is quadratically stable in the large if there exists a symmetric matrix \( P \) such that

\[
P > 0 \tag{7}
\]

\[
\Lambda_{ii}^T P + P \Lambda_{ii} < 0 \quad (i = 1, 2, \ldots, r) \tag{8}
\]

\[
\Lambda_{ij}^T P + P \Lambda_{ij} \leq 0 \quad (1 \leq i < j \leq r, \alpha_i (\mathbf{x}) \alpha_j (\mathbf{x}) \neq 0) \tag{9}
\]

It is noted that throughout this paper the origin \( \mathbf{x}(t) = 0 \) is assumed to be the only equilibrium point of the fuzzy control system.

### III. PROBLEM STATEMENT

Consider a general nonlinear system of the form,

\[
\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), u(t)) \tag{10}
\]

where \( f \in \mathbb{R}^n \) is a sufficiently smooth nonlinear function in the state \( \mathbf{x}(t) \in \mathbb{R}^n \) and control input \( u(t) \in \mathbb{R}^m \). Also, The following constraint is forced on the control input at all times \( t \geq 0 \).

\[
||u(t)|| \leq \mu \tag{11}
\]

As well known, such a nonlinear system like (10) can be effectively approximated in the fuzzy rule manner like (1) with some appropriate membership functions for fuzzy sets in both antecedent and consequent parts of the plant fuzzy rules. Then, the task left to us is to design \( r \) local linear state feedback law (4) such that the origin of the closed-loop system with the input constraint (11) is quadratically stable in the large. To this end, we have to transfer inequality (11) into an LMI form that eventually joins other LMIs like (7)-(9) for the system stability. In addition, the LMI corresponding to (11) leads to a bounded initial state condition. Actually, as mentioned later on, the initial state condition is very conservative. Therefore, we have to ease the condition to let the number of initial state satisfying the condition becomes as large as possible.

On the other hand, from Theorem 1 we can see that the stability of the fuzzy control system is reduced to a problem of finding a common \( P \). Obviously, if the number of fuzzy rules, \( r \), is large, it might be difficult to find a common \( P \) satisfying the LMIs conditions (7)-(9) in the theorem. Therefore, the LMIs conditions are conservative. One of aims in this paper is to give some more relaxed LMIs conditions that guarantee the stability of the fuzzy control system and is suitable for the design of fuzzy controller.

### IV. CONTROLLER DESIGN AND STABILITY ANALYSIS

First, we give a proposition which is closely related to the system design [3].

**Proposition 1:** Assume that the initial state condition \( \mathbf{x}(0) \) is known. The constraint \( ||u(t)|| \leq \mu \) is enforced at all times \( t \geq 0 \) if the LMIs

\[
\begin{bmatrix}
1 \\
\mathbf{x}(0)
\end{bmatrix}^T P^{-1} \geq 0, \tag{12}
\]

\[
\begin{bmatrix}
P^{-1} \\
F_i P^{-1} (P^{-1})^T F_i^T
\end{bmatrix} \geq 0 \tag{13}
\]

\[
(i = 1, 2, \ldots, r)
\]

hold.

By combining Theorem 1 with Proposition 1, the next theorem follows.

**Theorem 2:** For the plant expressed by fuzzy rule (1) subject to input constraint (11), and given feedback control gain matrix \( F_i \) for \( i = 1, 2, \ldots, r \), the equilibrium of the continuous fuzzy control system is quadratically stable in the large under linear state feedback control law (4) if there exists a symmetric positive matrix \( P \in \mathbb{R}^{n \times n} \) such that the following matrix inequalities hold:

\[
P > 0 \tag{14}
\]

\[
\Lambda_{ii}^T P + P \Lambda_{ii} < 0 \quad (i = 1, 2, \ldots, r) \tag{15}
\]

\[
\Lambda_{ij}^T P + P \Lambda_{ij} \leq 0 \quad (1 \leq i < j \leq r, \alpha_i \alpha_j \neq 0) \tag{16}
\]

\[
\begin{bmatrix}
1 \\
\mathbf{x}(0)
\end{bmatrix}^T P^{-1} \geq 0, \tag{17}
\]

\[
\begin{bmatrix}
P^{-1} \\
F_i P^{-1} (P^{-1})^T F_i^T
\end{bmatrix} \geq 0 \tag{18}
\]

\[
(i = 1, 2, \ldots, r)
\]

**Proof:** If a Lyapunov function candidate is chosen as \( V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x} \), the time derivative of \( V(\mathbf{x}) \) along the solution trajectories is as follows.

\[
\dot{V} = \dot{\mathbf{x}}^T P \mathbf{x} + \mathbf{x}^T P \dot{\mathbf{x}} \tag{19}
\]

Substituting (14)-(16) into (19), we get

\[
\dot{V}(t) \leq 0 \tag{20}
\]

It means that the system is quadratically stable in the large, and

\[
\mathbf{x}(t)^T P \mathbf{x}(t) \leq \mathbf{x}(0)^T P \mathbf{x}(0) \tag{21}
\]

As the proof of Proposition 1 [3], inequalities (17), (18) and (21) lead to \( ||u|| \leq \mu^2 \). This completes the proof.

The stability conditions presented in Theorem 2 can be used to obtain an LMI-based design for the feedback matrix.
gain \( F_i \ (i = 1, 2, \ldots, r) \) in (4) of the controlled plant given by (1) to be quadratically stable in the large.

**Theorem 3:** For the plant expressed by fuzzy rule (1) subject to input constraint (11), the closed-loop system is quadratically stable in the large under linear state feedback control law (4) if there exist symmetric positive matrix \( Q \in \mathbb{R}^{n \times n} \) and matrices \( M_i \in \mathbb{R}^{n \times n} \ (i = 1, 2, \ldots, r) \) such that the following matrix inequalities hold:

\[
Q > 0
\]
\[
QA_i^T + A_iQ - M_i^T B_i^T - B_i M_i < 0 \quad (i = 1, 2, \ldots, r)
\]
\[
Q A_j^T + A_j Q + Q A_j^T + A_j Q - M_j^T B_j^T - B_j M_j \leq 0 \quad (1 \leq i < j \leq r, \alpha_i \alpha_j \neq 0)
\]

**Proof:** If we let \( Q = P^{-1} \), and \( M_i = F_i Q_i \), it is easy to check that LMIs (14)-(18) in Theorem 2 are equivalent to (22)-(26) in Theorem 3, respectively. Therefore this theorem is proven.

When these LMIs are feasible, the feedback matrix gain is given by

\[
F_i = M_i Q^{-1} \quad (i = 1, 2, \ldots, r)
\]

**V. EASING CONSERVATISM**

### A. The largest ellipsoid

In this paper, in order to ensure that the control input satisfies the input constraint (11), LMI (25) is necessary to be held. Actually, this condition is very conservative. By Schur complement LMI (25) is equivalent to

\[
x(0)^T Q^{-1} x(0) \leq 1
\]

Define an ellipsoid

\[
E(Q) = \{x \in \mathbb{R}^n \mid x^T Q^{-1} x \leq 1\}
\]

then \( x_0 \) in \( E(Q) \) satisfies LMI (25). Obviously, a larger ellipsoid leads to a less conservative LMI.

**Example 1.** Consider a second order system, i.e., \( x \in \mathbb{R}^2 \), with an initial state \( x_0 = [0.2, 0.6] \). Given two \( Q \)s

\[
Q_1^{-1} = \begin{bmatrix}
2.0051 & -0.2590 \\
-0.2590 & 1.6854
\end{bmatrix},
\]

\[
Q_2^{-1} = \begin{bmatrix}
6.0152 & -0.7770 \\
-0.7770 & 5.0562
\end{bmatrix}
\]

in (25) where \( Q_1^{-1} < Q_2^{-1} \), the ellipsoids are depicted in Fig.1 in which \((x_{10}, x_{20})\) denotes the initial state.

From the example above, we can see that for a given initial state \( x(0) \), LMI (25) is satisfied with \( Q_1 \), but is not satisfied with \( Q_2 \) where \( Q_1 > Q_2 \). It means that we should find the largest \( Q \), which leads to the largest \( E(Q) \) in order to let \( x(0) \) satisfy the LMI as easy as possible.

To make the ellipsoid (29) larger as possible, clearly, it is necessary to let \( Q \) be maximum. Here we adopt the idea of reference set in [9]. Let \( \mathcal{X}_R \subset \mathbb{R}^n \) be a prescribed bounded convex set containing origin. For the ellipsoid \( E(Q) \), define

\[
\alpha_R = \sup \{ \alpha > 0 \mid \alpha \mathcal{X}_R \subset E(Q) \}.
\]

If \( \alpha_R > 1 \), then \( \mathcal{X}_R \subset E(Q) \). Therefore it is necessary to find the largest \( \alpha_R \) i.e.,

\[
\sup_Q \alpha \quad s.t. \quad \alpha \mathcal{X}_R \subset E(Q)
\]

The ellipsoid in (31) with \( \alpha_R \) is the maximum. There are two typical types of \( \mathcal{X}_R \): one is an ellipsoid

\[
\mathcal{X}_R = \{x \in \mathbb{R}^n \mid x^T R^2 x \leq 1\}
\]

where \( R > 0 \) which can be chosen, for example, as an identity matrix with compatible dimension, and the other one is a polyhedron

\[
\mathcal{X}_R = \text{conv \{ } x_1^2, x_0^2, \ldots, x_r^2 \text{ } \}
\]

where \text{conv} denotes convex hull, and \( x_0^1, x_0^2, \ldots, x_0^r \) are \textit{a priori} given points in \( \mathbb{R}^n \).

Now we transform optimization constrain (31) into an LMI. If the shape reference set is given by the ellipsoid (32), then \( \alpha \mathcal{X}_R \subset E(Q) \) in (31) is equivalent to

\[
\frac{R}{\alpha^2} \geq Q^{-1}
\]

It can be rewritten as an LMI form as follows.

\[
R^{-1} \leq Q, \quad \text{or} \quad \begin{bmatrix}
\gamma R & I \\
I & Q
\end{bmatrix} \geq 0
\]

where \( \gamma = \frac{1}{\alpha^2} \). Obviously, the maximum value of \( \alpha \) leads to the minimum value of \( \gamma \), which means that finding the maximum value of \( \alpha \) in (31) can be replaced with finding the minimum value of \( \gamma \) when using an LMI form in (35) instead of the constraint in (31).

If the shape reference set is given by the hyphehedron (33), then \( \alpha \mathcal{X}_R \subset E(Q) \) in (31) is equivalent to

\[
\alpha^2 (x_0^i)^T P x_0^i \leq \rho, \quad i = 1, 2, \ldots, l
\]
It can be rewritten as an LMI form as follows.
\[
\begin{bmatrix}
\gamma & (x_0^T) \\
\end{bmatrix}
\begin{bmatrix}
x_0^T \\
\end{bmatrix} \geq 0
\tag{37}
\]

**Example 2.** Consider a closed-loop system with input constraint \(|u| \leq \mu = 10\),
\[
A = \begin{bmatrix}
0.8876 & -0.5555 \\
0.5555 & 1.5542 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0.1124 \\
0.5555 \\
\end{bmatrix},
\]
and an initial state \(x_0 = [0.2127, 0.2475]^T\). The ellipsoid (32) with \(R = I\) is chosen as the shape reference set. In order to solve LMIs (22)-(26), and (35), software package Robust Control Toolbox (MATLAB) is used. The solutions are
\[
\gamma^* = 0.8783, \quad Q_1^* = \begin{bmatrix}
4.2385 & 4.5300 \\
4.3000 & 7.7584 \\
\end{bmatrix}.
\]

In addition, without the optimization, solution for LMIs (22)-(26) is
\[
Q_2^* = \begin{bmatrix}
3.7256 & 4.0915 \\
4.0915 & 6.0244 \\
\end{bmatrix}.
\]

It is clear that \(Q_1^* > Q_2^*\). Fig.2 shows the graphs of \(x^T(Q_1^*)^{-1}x = 1\) and \(x^T(Q_2^*)^{-1}x = 1\), where we can confirm that ellipsoid \(E(Q_1^*)\) is larger than \(E(Q_2^*)\). It means that for a given initial state \(x(0)\), compared with \(Q_2^*\), LMI (25) with \(Q_1^*\) is easier to be satisfied, which leads to less conservative.

![Fig. 2. Illustration of \(x^T(Q_1^*)^{-1}x = 1\) and \(x^T(Q_2^*)^{-1}x = 1\)](image)

Therefore the following theorem can be summarized.

**Theorem 4:** For the plant expressed by fuzzy rule (1) subject to input constraint (11), the closed-loop system under linear state feedback control law (4) is quadratically stable in an ellipsoid \(E(Q)\) to be maximum, if there exist positive symmetric matrices \(P, R_{ij} (i, j = 1, 2, \ldots, r)\) such that the following LMIs optimization problem is feasible:
\[
\begin{align*}
\min_{Q, M_i} \gamma \\
\text{s.t.} \quad \text{LMIs (22) – (26)} \\
\text{LMI (35), or (37)}
\end{align*}
\tag{38}
\]

**B. More relaxed LMIs conditions**

In order to make the closed-loop system quadratically stable in the large and obtain the feedback matrix gains in (27), it requires that LMIs in (38) are feasible. Especially, there are more than \((r + 1)\) extra LMIs in Theorems 2-4 in comparison with the regular LMIs conditions in Theorem 1. In other words, the LMIs conditions in the theorems are very conservative, and we have to relax the LMIs conditions to make the LMIs more feasible.

As far as we know, the method proposed by Teixeira et al. [6] is most effective to relax the LMIs conditions. However, from our repeated computer simulations we observed that, one of some extra terms played more important role than the others in relaxing the LMIs conditions in the method. The method shown below is based on the method proposed by Teixeira et al. [6].

**Theorem 5:** The equilibrium of the continuous fuzzy control system (6) is quadratical stable in the large if there exist symmetric matrices \(P, R_{ij} (i, j = 1, 2, \ldots, r)\) subject to \(j > i\) such that the following inequalities hold:
\[
P > 0
\tag{39}
\]
\[
\begin{bmatrix}
\hat{Q}_{11} \quad -Z_{1h} & \cdots & \cdots & \hat{Q}_{1r} \\
-Z_{21h} \quad \hat{Q}_{22} \cdots & \cdots & \cdots \\
\vdots \quad \vdots \quad \ddots & \cdots & \hat{Q}_{rh} \\
-Z_{r1} \quad \cdots & \cdots & \cdots & -Z_{rr} \\
\end{bmatrix} < 0
\tag{40}
\]

where
\[
\begin{align*}
\hat{Q}_{ij} &= \Lambda_{ij}^T P + P \Lambda_{ij} \\
\Xi_{ij} &= \begin{cases} 
\hat{Q}_{ij} + \frac{1}{2} W_{ij}, & \text{if } i < j \\
\hat{Q}_{ji} + \frac{1}{2} W_{ij}, & \text{if } i > j \\
0, & \text{if } i = j
\end{cases} \\
W_{ij} &= \begin{cases} 
R_{ij}, & \text{if } i = j \\
0, & \text{if } i \neq j
\end{cases}
\end{align*}
\tag{41}
\]

**Proof:** The proof is similar to the proof of Theorem 5 in the paper [6]. Here we omit the proof.

**Example 3.** This example demonstrate the effectiveness of the relaxed LMIs conditions. Consider \(r = 2\),
\[
A_1 = \begin{bmatrix}
2 & -10 \\
1 & 0 \\
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 \\
0 \\
\end{bmatrix},
\]
\[
A_2 = \begin{bmatrix}
a & -10 \\
1 & 3 \\
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
b \\
0 \\
\end{bmatrix}.
\]

The local feedback matrix gains \(F_1\) and \(F_2\) are determined by selecting \([-2 - 2]\) as the eigenvalues of the subsystems in the PDC. While the gray area in Fig.3 is the feasible area satisfying the LMIs in Theorem 1, the gray area in Fig.4 is the feasible area satisfying the LMIs in Theorem 5. Clearly, it is found in these figures that Theorem 5 leads to a less conservative result.

Note that in comparison with the method proposed in [6], Theorem 5 is relatively simple in terms of two matrix variables to be omitted. However, the effectiveness for relaxing
the LMIs conditions is almost same with the method in [6]. Obviously, the simpler, the better.

\[ Q_{ij} = \frac{1}{2} \left( Q A_i^T + A_i Q + Q A_j^T + A_j Q - M_j^T B_i^T - B_i M_j - M_i^T B_j^T - B_j M_i \right) \] (45)

VI. SIMULATION

Consider the problem of balancing and swing-up of an inverted pendulum on a cart. The equations of motion for the pendulum are

\[ \dot{x}_1 = x_2 \] (46)

\[ \dot{x}_2 = \frac{g \sin(x_1) - \frac{\alpha m x_2^2 \sin(2x_1)}{2} - \mu g \cos(x_1) v}{\frac{b}{2} - \alpha m} \] (47)

where \( x_1 \) denotes the angle of the pendulum from the vertical, \( x_2 \) is the angular velocity, \( g = 9.8 m/s^2 \) is the gravity constant, \( m \) is the mass of the pendulum, \( M \) is the mass of the cart (in Kilo-Newton), \( a = \frac{1}{m+M} \) and \( \mu = 1000 \). We choose \( m = 2.0 kg, M = 8.0 kg, l = 0.5 m \) in this simulation.

The control objective is to balance the inverted pendulum with actuator saturation \( \bar{u} = 1.0 \). Here we follow [12] to approximate the system by the following two-rule fuzzy model:

Rule 1 : IF \( x_1 \) is \( \bar{0} \) THEN \( \dot{x} = A_1 x + B_1 u \)

Rule 2 : IF \( x_1 \) is \( \pm \pi / 2 \) THEN \( \dot{x} = A_2 x + B_2 u \)

where

\[ A_1 = \begin{bmatrix} 0 & \frac{g}{\frac{b}{2} - \alpha m} \\ \frac{\alpha m \beta^2}{\frac{b}{2} - \alpha m} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \frac{\mu g \cos(x_1)}{\frac{b}{2} - \alpha m} \end{bmatrix} \]

\[ A_2 = \begin{bmatrix} 0 & \frac{g}{\frac{b}{2} - \alpha m} \\ \frac{\alpha m \beta^2}{\frac{b}{2} - \alpha m} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{\mu g \cos(x_1)}{\frac{b}{2} - \alpha m} \end{bmatrix} \]

and \( \beta = \cos(86^\circ) \). Membership functions for fuzzy sets \( \bar{0} \) and \( \pi / 2 \) are chosen as follows.

\[ \mu_{\bar{0}}(x_1) = \begin{cases} \cos(x_1), & -\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \] (48)

\[ \mu_{\bar{\pi}/2}(x_1) = \begin{cases} 1 - \cos(x_1), & -\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \] (49)

In order to make a comparison between the regular LMIs conditions (Theorem 3) and the relaxed LMIs conditions (Theorem 6), first, we solve (44) in Theorem 6 and obtain

\[ F_1 = [-0.2697, -0.1734], \quad F_2 = [-0.4660, -0.2364] \] (50)

where \( F_i = M_i * Q^{-1} \) (45), \( i = 1, 2 \), \( M_1 = [-60.4328, -135.3235], \) and \( M_2 = [-147.3167, -117.7830] \), and

\[ Q^* = \begin{bmatrix} 108.1508 & -224.6436 \\ -224.6436 & 489.1824 \end{bmatrix} \]
with $\gamma = 0.2433$. On the other hand, by solving LMIs (22)-(26) in Theorem 3 without optimization, it also feasible along with

$$Q = \begin{bmatrix} 13.6960 & -21.8321 \\ -21.8321 & 44.0223 \end{bmatrix}.$$ 

The eigenvalues of $(Q^* - Q)$ are 1.7010 and 537.9139 indicating that $Q^* > Q$. Fig.5 shows the graphs of $x^T(Q^*)^{-1}x = 1$ and $x^TQ^{-1}x = 1$ where the domain of attraction involving with $Q^*$ is much larger than the one with $Q$. From the domains in Fig.5 we can confirm that the both are bigger than $|x_1| = \pi/2$, which means that $F$ obtained by Theorem 6 or 3 can theoretically balance the pendulum even with initial condition that $x_{10} \geq 90^\circ$. However, as expressed in the membership functions in (48) and (49), it is not necessary to consider cases that $x_1 \geq \pi/2 \text{ rad} = 90^\circ$ due to the physical situation. In this simulation, we consider three cases for the initial state: $x_{10} = 80^\circ, 80^\circ, 50^\circ$ with $x_{20} = 0$. The system responses controlled by $F$ in (50) are shown in Fig.6 where the solid line is with $x_{10} = 80^\circ$, the dashed line with $x_{10} = 80^\circ$, and the dash-dotted line with $x_{10} = 50^\circ$. As mentioned before, the limited amplitude for actuator is 1.0. The control inputs for the three cases are depicted in Fig.7, where it is clear that they are all less than the limitation.

![Fig. 5. The domain of attractions obtained by Theorem 3 and 6](image)

![Fig. 6. State $x_1$ and $x_2$ with different initial conditions: $80^\circ$, $80^\circ$ and $50^\circ$.](image)

![Fig. 7. Control inputs with different initial conditions: $80^\circ$, $80^\circ$ and $50^\circ$.](image)

### VII. CONCLUSION

This paper dealt with stability analysis and control design for a class of nonlinear systems with input constraint when using T-S fuzzy model. After choosing the control design to LMIs approach, the attention was paid to how to make the LMIs conditions less conservative. Also, a few examples were given to illustrate the effectiveness of relaxed LMIs conditions.

### REFERENCES


